Regression III

Feature Selection and Regularization

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Goals for Today

1. Discuss feature selection and its relationship with more conventional model testing and discrimination.
2. Develop all subsets regressions and consider a comparison of models.
3. Describe cross-validation and its utility for helping choose tuning parameters.
4. Discuss regularization and regularized regression models - Ridge regression, the LASSO, Elastic Net and Adaptive LASSO.
    - Consider how these models adjudicate collinearity problems.
5. Consider the problem of post-selection inference.

Feature Selection

Sometimes, we may not want have a couple of different models; instead, we have a bunch of variables and we want to find out which ones are "important".

- Important, in this case, means predictive power - ability to capture variation or discriminate among values in the dependent variable.
- Feature selection automate the process of choosing features based on what "works" in the data.

Some questions you might have:

- Q: Isn't this atheoretical? A: Yes
- Q: Isn't this data mining? A: Yes
- Q: Isn't this kind of analysis disingenuous? A: It depends.

Notes

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Feature Selection: Manually

How many of you have written a paper where you had a theory, that theory produced a single model specification, the operationalization of the concepts in measures was utterly uncontroversial and the empirical model was so thoroughly beyond reproach and obviously useful that no diagnostics were needed?

- We generally use **ad hoc** methods of feature selection.
- These are only slightly less problematic - more on volume than principle.
- If we're going to use the data to select features, why not go all the way?

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Subset Methods

- The goal of subset methods is to examine *which subsets give the best fit to the data for a given number of predictors*
- Even when the number of variables is large, it is feasible to examine all subsets
  - If there are p potential predictors, then there are 2^p possible models
- Subset techniques have the advantage over stepwise regression of revealing alternative *nearly equivalent* models and thus avoid the appearance of a uniquely "correct" result
- Several measures can be used to determine the best model subset
  - R^2
  - AIC
  - BIC
  - Mallow's C_p-statistic

Notes

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Mallow's Cp Statistic

Mallow's C_p-statistic is defined as:

$$\begin{aligned} C_p &= \frac{\sum E_i^2}{S_E^2} + 2p - n \\ &= (K + 1 - p)(F_p - 1) + p \end{aligned}$$

- S_E^2 is for the full model containing k explanatory variables; RSS $(\sum E_i^2)$ is from the subset model with p explanatory variables
- F_p is the incremental F-test for the hypothesis that the regressors omitted from the subset have slope 0. If the hypothesis is true, $E(F_p) \simeq 1$, and thus $C_p \simeq p$
- C_p increases with the residual sum of squares.
- A good model, then, has C_p as close to p as possible
- A plot of C_p against p allows us to choose the model

Notes

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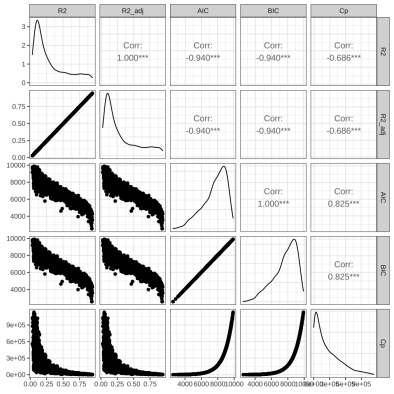
Comparison



Notes

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Model Selection Example: Ericksen Data (1)

```
library(leaps)
library(car)
Ericksen <- DAMisc::scaleDataFrame(Ericksen)
X <- model.matrix(undercount ~ .,
  data=Ericksen)[,-1]
y <- model.response(model.frame(undercount
   ~ ., data=Ericksen))
rmods <- regsubsets(x=X, y=y, method="exhaustive",
    all.best=TRUE, nbest=10)
```


Notes

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Model Selection Example: Ericksen Data (3)

- Subset selection is implement in R using two packages: `leaps` and `car`
- Using the `regsubsets` function, you specify the full model and how many subsets you want
- The `subsets` function in `car` graphs the models with the subset size on the horizontal axis and the statistic used for fit on the vertical axis
- The `subsets` function allows you to specify the following statistics: Mallows C_p `cp`, R^2 `rsq`, adjusted R^2 `adjrs2`, RSS `rss` or BIC `bic`
- You can also specify the number of predictors you want in the model (below specifies 3 to 5 predictors)

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Subsets plot for the Ericksen Data

```
library(car)
subsets(rmods, statistic="cp", legend=F)
```


Notes

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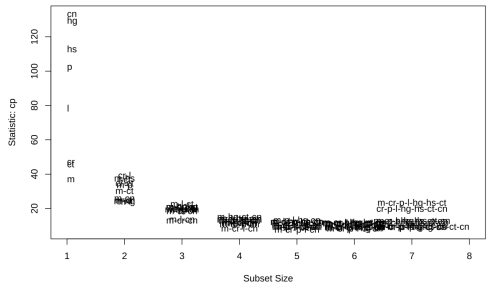
Subsets plot for the Ericksen Data (2)

```
library(car)
subsets(rmods, statistic="cp", legend=F)
```



Notes

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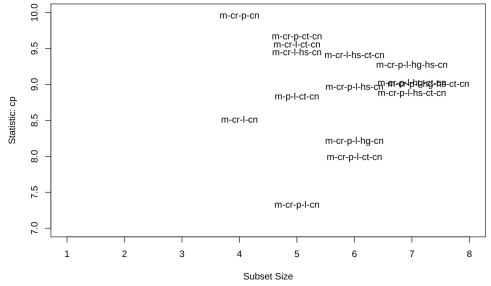
Subsets plot zoomed in



Notes

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$C_p - K$

We could also consider the models that have the smallest $C_p - K$.

```
s <- summary(rmods)
K <- rowSums(s$which)
abbrevs <- c("m", "cr", "p", "l", "hg", "hs", "ct", "cn")
mod <- apply(s$which[,-1], 1, function(i)
  paste(abbrevs[which(i)], collapse="-"))
dat <- tibble(
  K=K,
  Cp = s$cp,
  adjr2 = s$adjr2,
  mod = mod,
  diff = Cp-K)
sub <- dat %>%
  filter(K < 9) %>%
  slice_min(diff, n=10)
```

```
## # A tibble: 10 x 5
##        K    Cp adjr2 mod                 diff
##    <dbl> <dbl> <dbl> <chr>              <dbl>
##  1     8  8.87 0.662 m-cr-p-l-hs-ct-cn  0.874
##  2     7  7.98 0.661 m-cr-p-l-ct-cn     0.983
##  3     8  9.01 0.661 m-cr-p-l-hg-ct-cn  1.01
##  4     7  8.21 0.660 m-cr-p-l-hg-cn     1.21
##  5     8  9.27 0.659 m-cr-p-l-hg-hs-cn  1.27
##  6     6  7.32 0.659 m-cr-p-l-cn        1.32
##  7     7  8.96 0.656 m-cr-p-l-hs-cn     1.96
##  8     7  9.41 0.653 m-cr-l-hs-ct-cn    2.41
##  9     6  8.83 0.651 m-p-l-ct-cn        2.83
## 10     8 11.4  0.647 m-cr-l-hg-hs-ct-cn 3.41
```


Notes

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Overfit Much?

How do we know we're not overfitting our data?

- Sometimes it's obvious - it's hard to argue that you're overfitting when your R^2 is 0.03.
- Generally, we don't know.

Cross-validation is a way of trying to protect us against overfitting the model.

Notes

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Cross-Validation (1)

If no two observations have the same Y, a p-variable model fit to $p + 1$ observations will fit the data precisely

- Of course, this will lead to biased estimators that are likely to give quite different predictions on another dataset (generated with the same DGP)
- Model validation allows us to assess whether the model is likely to predict accurately on future observations or observations not used to develop this model
  - External validation involves retesting the model on new data collected at a different point in time or from a different population
  - Internal validation (or cross-validation) involves fitting and evaluating the model carefully using only one sample

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Cross-Validation (2)

Cross-validation is similar to bootstrapping in that it resamples from the original data

- The basic form involves randomly dividing the sample into two subsets:
  - The first subset of the data (screening sample) is used to select or estimate a statistical model
  - The second subset is then used to test the findings
- Can be helpful in avoiding capitalizing on chance and over-fitting the data - i.e., findings from the first subset may not always be confirmed by the second subsets
- Cross-validation is often extended to use several subsets (either a preset number chosen by the researcher or leave-one-out cross-validation)

Notes

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Cross-Validation (3)

- The data are split into k subsets (usually $3 \leq k \leq 10$)
- Each of the subsets are left out in turn, with the regression run on the remaining data
- Prediction error is then calculated as the sum of the squared errors:

$$RSS = \sum (Y_i - \hat{Y}_i)^2$$

- We choose the model with the smallest average "error"

$$MSE = \frac{\sum (Y_i - \hat{Y}_i)^2}{n}$$

- We could also look to the model with the largest average R^2

Notes

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Cross-Validation (4)

How many observations should I leave out from each fit?

- There is no rule on how many cases to leave out, but Efron (1983) suggests that grouped cross-validation (with approximately 10% of the data left out each time) is better than leave-one-out cross-validation

Number of repetitions

- Harrell (2001:93) suggests that one may need to leave $\frac{1}{10}$ of the sample out 200 times to get accurate estimates

Cross-validation does not validate the complete sample

- External validation, on the other hand, validates the model on a new sample
- Of course, limitations in resources usually prohibits external validation in a single study

Notes

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Cross-Validation in R

```
library(boot)
dat <- read.csv("http://www.quantoid.net/files/reg3/weakliem.txt"
  header=T)
dat <- dat[-c(25,49), ]
mod1 <- glm(secpay ~ poly(gini, 3)*democrat, data=dat)
mod2 <- glm(secpay ~ gini*democrat, data=dat)

deltas <- NULL
for(i in 1:25){
deltas <- rbind(deltas, c(
  cv.glm(dat, mod1, K=5)$delta,
  cv.glm(dat, mod2, K=5)$delta)
)}
out <- matrix(colMeans(deltas), ncol=2)
rownames(out) <- c("delta_1", "delta_2")
colnames(out) <- c("Model 1", "Model 2")
```

```
##         Model 1 Model 2
## delta_1  0.0212  0.0058
## delta_2  0.0180  0.0057
```

The `delta_1` term is the average raw cross-validation error. The `delta_2` term corrects for using k-fold rather than leave-one-out CV.

Notes

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Tidy CV

```
cv_fun <- function(split, ...){
  u1 <- update(mod1, data=analysis(split))
  u2 <- update(mod2, data=analysis(split))
  e1_sq <- (assessment(split)$secpay -
              predict(u1, newdata=assessment(split)))^2
  e2_sq <- (assessment(split)$secpay -
              predict(u2, newdata=assessment(split)))^2
  tibble(
    err = c(sum(e1_sq), sum(e2_sq)),
    n = c(length(e1_sq), length(e2_sq)),
    model = factor(c("poly_int", "linear_int")))
}
library(purrr)
library(rsample)
v <- vfold_cv(dat, v=5, repeats=250) %>%
  mutate(err = map(splits, cv_fun))
```

```
v %>% unnest(err) %>%
  group_by(id, model) %>%
  summarise(mse = sum(err)/sum(n)) %>%
  pivot_wider(names_from="model", values_from = "mse") %>%
  mutate(diff = poly_int - linear_int) %>%
  ungroup %>%
  summarise(across(c(linear_int, poly_int), mean),
            "p(MSE_M1 > MSE_M2)" = mean(diff > 0))
```

```
## # A tibble: 1 x 3
##   linear_int poly_int `p(MSE_M1 > MSE_M2)`
##        <dbl>    <dbl>                <dbl>
## 1    0.00578   0.0103                   1
```


Notes

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Cross-validating Span in Loess

We could use cross-validation to tell us something about the span in our Loess model.

- First, split the sample into K groups (usually 10).
- For each of the $k = 10$ groups, estimate the model on the other 9 and get predictions for the omitted groups observations. Do this for each of the 10 subsets in turn.
- Calculate the CV error: $\frac{1}{n}\sum(y_i - \hat{y}_i)^2$
- Potentially, do this lots of times and average across the CV error.

```
set.seed(1)
n <- 400
x <- 0:(n-1)/(n-1)
f <- 0.2*x^11*(10*(1-x))^6+10*(10*x)^3*(1-x)^10
y <- f + rnorm(n, 0, sd = 2)
tmp <- data.frame(y=y, x=x)
lo.mod <- loess(y ~ x, data=tmp, span=.75)
```


Notes

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Minimizing CV Criterion Directly

```
library(DAMisc)
cvlo <- DAMisc:::cv.lo2
best.span <- optimize(cvlo, c(.05,.95), form=y ~ x, data=tmp,
         numiter=5, K=10)
best.span
```

```
## $minimum
## [1] 0.2611968
##
## $objective
## [1] 3.914841
```

There is also a canned function in `fANCOVA` that optimizes the span via AICc or GCV.

```
library(fANCOVA)
best.span2 <- loess.as(tmp$x, tmp$y, criterion="aicc")
best.span2$pars$span
```

```
## [1] 0.2136455
```

```
best.span3 <- loess.as(tmp$x, tmp$y, criterion="gcv")
best.span3$pars$span
```

```
## [1] 0.1483344
```


Notes

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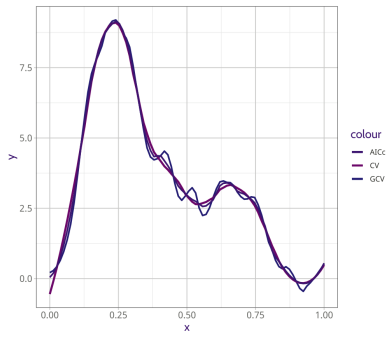
The Curve



Notes

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Manually Cross-Validating λ in the Y-J Transform

Sometimes, optimizing the cross-validation criterion fails.

- Randomness in the CV procedure can produce a function that has several local minima.
- You could force the same random split at every evaluation by hand-coding the CV, but this might not be the best idea.
- If optimization of the CV criterion fails, you could always do it manually.

Notes

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Example

```
cvoptim_yj <- function(pars, form, data, trans.vars, K=5, numiter=10){
    require(boot)
      require(VGAM)
    form <- as.character(form)
    for(i in 1:length(trans.vars)){
        form <- gsub(trans.vars[i], paste("yeo.johnson(", trans.vars[i],
        ",", pars[i], ")", sep=""), form)
    }
    form <- as.formula(paste0(form[2], form[1], form[3]))
    m <- glm(as.formula(form), data, family=gaussian)
    d <- lapply(1:numiter, function(x)cv.glm(data, m, K=K))
    mean(sapply(d, function(x)x$delta[1]))
}

lams <- seq(0,2, by=.1)
s <- sapply(lams, function(x)cvoptim_yj(x, form=prestige ~ income + education + women,
    data=Prestige, trans.vars="income", K=3))
ggplot() +
  geom_line(mapping=aes(y=s, x=lams)) +
  theme_bw() +
  mytheme() +
  labs(x="Lambda", y="CV Error")
```


Notes

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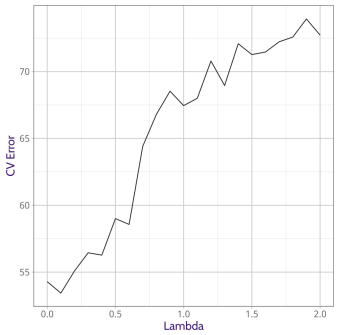
Figure



Notes

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Shrinkage Estimators

Shrinkage estimators can reduce sampling variability and sometimes improve model fit (particularly in the presence of collinearity).

- Shrinkage estimators impose constraints on the fitted model (particularly on the size of the coefficients).
- The result of these constraints is to shrink the estimates toward zero.
- Ridge Regression and the LASSO are the two most prominent shrinkage estimators.

NB: these are *biased* estimators, so they might be good for stabilizing predictions, but they won't be particularly good for more conventional theory testing.

Notes

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Ridge Regression

Ridge Regression minimizes the following function:

$$\sum_{i=1}^{N}\left(y_i - \beta_0 + \sum_{j=1}^{p}\beta_j x_{ij}\right)^2 + \lambda\sum_{j=1}^{p}\beta_j^2$$

- λ is a tuning parameter that governs the relative of RSS and the penalty on fitting the regression surface.
- As $\lambda \to 0$, the estimates get increasingly close to the OLS estimates.
- As $\lambda \to \infty$, the estimates get increasingly close to zero.

The choice of λ is important and is often done with cross-validation.

Notes

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CV MSE

```
library(glmnet)
library(ggplot2)
library(rio)
library(tidyr)
banks99 <- import(
  "http://quantoid.net/files/reg3/banks99.dta")
banks99s <- scaleDataFrame(banks99[,-c(1,2,4)])
X <- scale(model.matrix(gdppc_mp ~. , data=banks99s))[,-1]
y <- model.response(model.frame(gdppc_mp ~. , data=banks99s))

library(glmnet)
loglam <- seq(6.8, -5, length=100)
g1 <- glmnet(X, y, alpha=0)

rcv <- cv.glmnet(X, y, alpha=0, lambda=exp(loglam))
plot(rcv)
```


Notes

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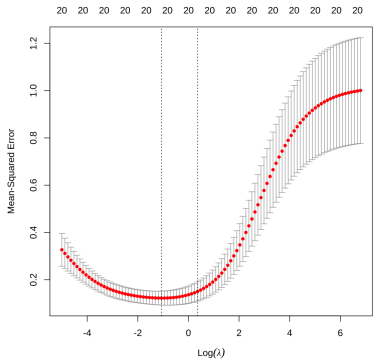
CV MSE (2)



Notes

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CV with Ridge Regression

```
r <- glmnet(X, y, alpha=0, lambda=exp(loglam))
ridge.mod <- glmnet(X,y, alpha=0, lambda=rcv$lambda.min)
mod <- lm(y ~ X)

l2o <- sqrt(sum(coef(mod)^2))
l2r <- apply(r$beta, 2, function(x)sqrt(sum(x^2)))
br <- r$beta %>% as.matrix %>% t %>% as.data.frame
br$ratio <- l2r/l2o
br <- br %>% pivot_longer(under5_mort:all_veh_pc, names_to="variable", values_to="coef")
ggplot(br, aes(x=ratio, y=coef)) +
  geom_line() +
  geom_vline(xintercept=(l2r/l2o)[87], lty=2) +
  geom_hline(yintercept=0, linetype=3) +
  facet_wrap(~variable) +
  theme_bw() +
  mytheme(panel.grid=element_blank()) +
  labs(x="Ratio of L2(ridge)/L2(OLS)", y="Coefficient")
```


Notes

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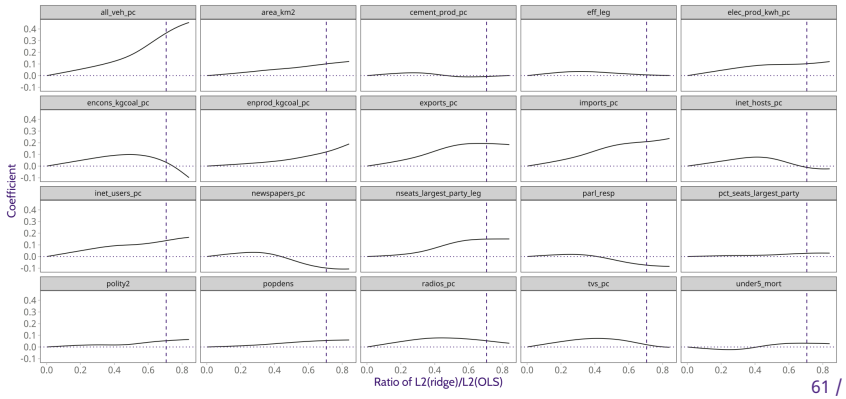
Plot



Notes

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Collinearity

```
set.seed(1234)
Sig <- diag(5)
Sig[3:5,3:5] <- .99
diag(Sig) <- 1
X <- MASS::mvrnorm(500,rep(0,5), Sig)
b <- c(1,1,1,0,0)
ystar <- X %*% b
y <- ystar + rnorm(500, 0, 2)
```

```
summary(m1 <- lm(y~ X))
```

```
##
## Call:
## lm(formula = y ~ X)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -5.4195 -1.3696 -0.0068  1.4012  5.3665
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept)  0.01686    0.08762   0.192   0.8475
## X1           1.17283    0.09359  12.532   <2e-16 ***
## X2           1.11657    0.09163  12.185   <2e-16 ***
## X3           1.21441    0.69342   1.751   0.0805 .
## X4           1.04118    0.68252   1.525   0.1278
## X5          -1.09301    0.70047  -1.560   0.1193
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 1.953 on 494 degrees of freedom
## Multiple R-squared:  0.469,    Adjusted R-squared:  0.4637
## F-statistic: 87.28 on 5 and 494 DF,  p-value: < 2.2e-16
```


Notes

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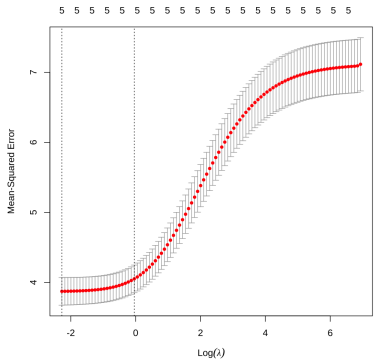
Collinearity (2)



Notes

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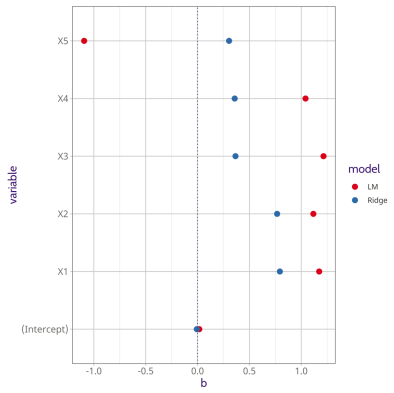
Collinearity (3)



Notes

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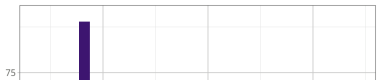
Prediction Variances

```
library(boot)
df <- cbind(y, X)
boot.ridge <- function(data, inds, ...){
  tmp <- data[inds,]
  y <- tmp[,1]
  X <- tmp[,-1]
  out <- glmnet(X,y,alpha=0, lambda=.8736)
  as.vector(coef(out))
}
br <- boot(statistic=boot.ridge,
           data=df, R=100)
v <- var(br$t)
pred.vars <- diag(cbind(1, X) %*%
                    v %*% t(cbind(1,X)))
lm.vars <- diag(cbind(1, X) %*%
                  vcov(m1) %*% t(cbind(1,X)))
```



Notes

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Predictions

```
summary(pred.vars/lm.vars)
```

```
##    Min. 1st Qu.  Median    Mean 3rd Qu.    Max.
##  0.1473  0.4181  0.5551  0.5619  0.6800  1.2081
```

```
ridge.preds <- cbind(1, X) %*% coef(r2)
lm.preds <- cbind(1,X) %*% coef(m1)
cor(as.vector(lm.preds), as.vector(ridge.preds))
```

```
## [1] 0.9847665
```


Notes

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LASSO (the L1 norm)

The LASSO (Least Absolute Shrinkage and Selection Operator) is another regularization method for estimating regression.

- Uses a different penalty than ridge regression:

$$\sum_{i=1}^{N}\left(y_i - \beta_0 + \sum_{j=1}^{p}\beta_j x_{ij}\right)^2 + \lambda\sum_{j=1}^{p}|\beta_j|$$

- Doesn't necessarily use all of the variables (i.e., some coefficients could be zero)
- Since not all variables are used in each fit, bootstrapping is more problematic here (though not impossible).

Notes

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The LASSO in R

```
banks99 <- import(
  "http://quantoid.net/files/reg3/banks99.dta")
banks99s <- scaleDataFrame(banks99[,-c(1,2,4)])
X <- scale(model.matrix(gdppc_mp ~. , data=banks99s))[,-1]
y <- model.response(model.frame(gdppc_mp ~. , data=banks99s))

loglam <- seq(6.8, -5, length=100)
cvg <- cv.glmnet(X,y, lambda=exp(loglam))
g <- glmnet(X, y, lambda=cvg$lambda.min)
```

```
round(cbind(coef(cvg), coef(mod)), 4)
```

```
## 21 x 2 sparse Matrix of class "dgCMatrix"
##                                s1
## (Intercept)              0.0000  0.0000
## under5_mort              .       0.0214
## area_km2                 .       0.1365
## inet_hosts_pc            .      -0.0032
## inet_users_pc            0.1022  0.1813
## enprod_kgcoal_pc         .       0.2801
## encons_kgcoal_pc         0.0125 -0.2730
## elec_prod_kwh_pc         .       0.1422
## cement_prod_pc           .       0.0073
## nseats_largest_party_leg .       0.1520
## eff_leg                  .      -0.0026
## pct_seats_largest_party  .       0.0250
## radios_pc                .       0.0140
## tvs_pc                   .      -0.0025
## newspapers_pc            .      -0.0930
## polity2                  .       0.0765
## parl_resp                .      -0.0853
## popdens                  .       0.0607
## imports_pc               0.1844  0.2825
## exports_pc               .       0.1673
## all_veh_pc               0.4832  0.5060
```


Notes

Type notes here...

Regularization path

```
r <- glmnet(X, y, alpha=0, lambda=exp(loglam))
g <- glmnet(X, y, alpha=1, lambda=exp(loglam))
mod <- lm(y ~ X)


br1 <- r$beta %>% as.matrix %>% t %>% as.data.frame
br2 <- g$beta %>% as.matrix %>% t %>% as.data.frame
br1$lambda <- br2$lambda <- loglam
br1 <- br1 %>% pivot_longer(under5_mort:all_veh_pc, names_to="variable", values_to="coef")
br2 <- br2 %>% pivot_longer(under5_mort:all_veh_pc, names_to="variable", values_to="coef")
br1$model <- factor(1, levels=c(1,2), labels=c("Ridge", "LASSO"))
br2$model <- factor(2, levels=c(1,2), labels=c("Ridge", "LASSO"))

br <- bind_rows(br1, br2)

ggplot(br, aes(x=lambda, y=coef, colour=model)) +
  geom_line() +
  facet_wrap(~variable, scales="free_y") +
  geom_vline(xintercept=log(rcv$lambda.min), col=pal2[1], lty=3) +
  geom_vline(xintercept=log(cvg$lambda.min), col=pal2[2], lty=3) +
  geom_hline(yintercept=0, linetype=3) +
  scale_colour_manual(values=pal2) +
  theme_bw() +
  mytheme(panel.grid=element_blank()) +
  labs(x="log(Lambda)", y="Coefficient")
```


Notes

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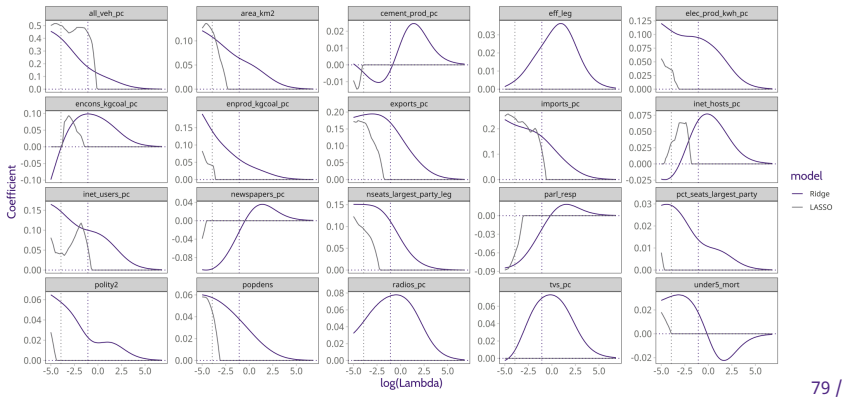
Plot



Notes

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Predictions

```
r1 <- glmnet(X, y, alpha=0, lambda=rcv$lambda.mi
g1 <- glmnet(X, y, alpha=1, lambda=cvg$lambda.mi
yhat <- mod$fitted.values
names(yhat) <- NULL

preds <- tibble(
  ols=yhat,
  ridge = as.vector(predict(r1, newx=X)),
  lasso= as.vector(predict(g1, newx=X))
)
```

```
ggpairs(preds) + mytheme() + theme_bw()
```

Notes

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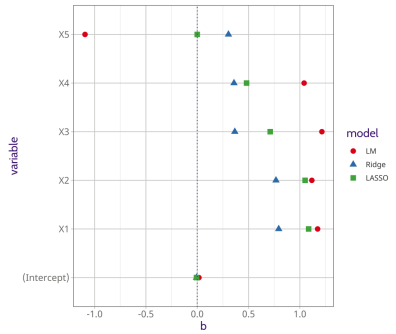
LASSO and collinearity

```
cvg2 <- cv.glmnet(scale(coll$X), coll$y, alpha=1
g2 <- glmnet(scale(coll$X), coll$y,
             alpha=1, lambda=cvg2$lambda.min)
r2 <- glmnet(scale(coll$X), coll$y,
             alpha=0, lambda=rcv2$lambda.1se)
coefs <- tibble(
  b = c(as.vector(m1$coef), as.vector(coef(r2)),
        as.vector(coef(g2))),
  model = factor(rep(1:3, each=length(coef(m1))),
           labels=c("LM", "Ridge", "LASSO")),
  variable = rep(names(m1$coef), 3))
p1 <- ggplot(coefs, aes(x=b, y=variable,
                        colour=model, shape=model)) +
  geom_point(size=3) +
  theme_bw() +
  scale_colour_manual(values=pal5[1:3]) +
  geom_vline(xintercept=0, lty=3)+
  mytheme()
```



Notes

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Elastic Net

The *Elastic Net* is a compromise between Ridge and LASSO regression:

$$\min_{\beta_0,\beta} \frac{1}{N} \sum_{i=1}^{N} w_i l(y_i, \beta_0 + \beta^T x_i) + \lambda \left[(1-\alpha)||\beta||_2^2/2 + \alpha||\beta||_1 \right],$$

- LASSO: $\alpha = 1$, Ridge: $\alpha = 0$
- α can be chosen a priori or you can experiment with several different values.
- Often setting α close to, but not exactly, 1 has nice properties.

Notes

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Elastic Net in Action

```
cv.enet <- list()
s <- seq(0.01, .99, length=25)
for(i in 1:length(s)){
    cv.enet[[i]] <- cv.glmnet(X, y, alpha = s[i])
}
cv.err <- sapply(cv.enet, function(x)min(x$cvm))
s[which.min(cv.err)]
```

```
## [1] 0.01
```

```
b <- sapply(cv.enet[c(1,7,25)],
    function(x)as.matrix(coef(x)))

plot.dat <- data.frame(
    b = c(b),
    group = as.factor(rep(c(.010,.255,.990),
        each = 21)),
    var = factor(rep(
        rownames(coef(cv.enet[[1]])), 3)))
library(ggplot2)
g <- ggplot(plot.dat, aes(x=b,
    y=reorder(var, b, mean))) +
    geom_point() +
    scale_colour_manual(values=pal3) +
    theme_bw() +
    facet_wrap(~group, nrow=1) +
    mytheme() +
    ylab("")
```


Notes

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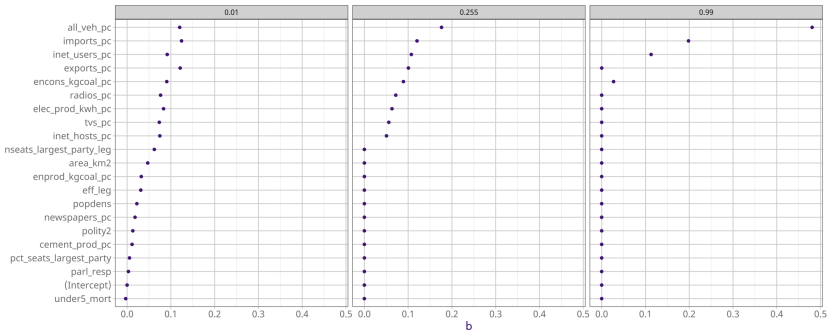
Figure



Notes

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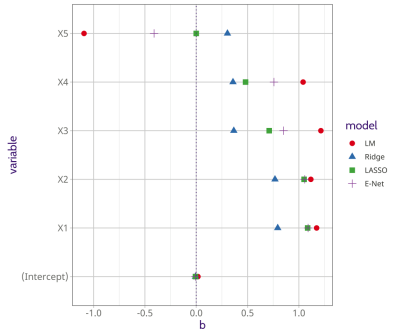
Elastic Net and Collinearity



Notes

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Adaptive Lasso

The lasso gives all variables the same penalty (λ). The adaptive lasso relaxes this assumption by allowing each parameter to have a different weight:

$$\underset{\beta}{\operatorname{argmin}} \left\| y - \sum_{j-1}^{p} \mathbf{x}_j \beta_j \right\|^2 + \lambda \sum_{j=1}^{p} w_j |\beta_j|$$

Where we use results from an auxiliary regression (OLS, Ridge or LASSO) to make the weights:

$$\hat{w}_j = \frac{1}{|\hat{\beta}_j|^\gamma}$$

γ is not usually estimated, but values 0.5, 1, and 2 are tried to evaluate sensitivity. The only technical constraint is that $\gamma > 0$. \

Notes

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Oracle Property

The Adaptive Lasso has been shown to have the Oracle property, that the selection procedure asymptotically chooses the right model:

- True 0 coefficients are estimated as 0 with probability that tends toward 1
- True non-zero coefficients are estimated as if the true sub-model were known.

Notes

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Steps for Adaptive LASSO

- Estimate the initial coefficients via regression model (OLS, Ridge or LASSO).
- Calculate the weights $w_j = \frac{1}{|\beta_j|^\gamma} \quad \gamma = \{0.5, 1, 2\}$.
- Use the weights as input to the LASSO routine.

Notes

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Adaptive LASSO

```
# estimate initial ridge regression and save coefficients
b.ridge <- coef(cv.glmnet(X,y, alpha=0))
# calculate weights
gamma <- 1
w.banks <- 1/(abs(b.ridge)^gamma)
# estimate the LASSO with the weights
cval <- cv.glmnet(X,y, penalty.factor=w.banks[-1])
```

```
coef(cval)
```

```
## 21 x 1 sparse Matrix of class "dgCMatrix"
##                                      s1
## (Intercept)              -1.059584e-17
## under5_mort               .
## area_km2                  .
## inet_hosts_pc             .
## inet_users_pc             .
## enprod_kgcoal_pc          .
## encons_kgcoal_pc          .
## elec_prod_kwh_pc          .
## cement_prod_pc            .
## nseats_largest_party_leg  .
## eff_leg                   .
## pct_seats_largest_party   .
## radios_pc                 .
## tvs_pc                    .
## newspapers_pc             .
## polity2                   .
## parl_resp                 .
## popdens                   .
## imports_pc                2.229577e-01
## exports_pc                .
## all_veh_pc                5.418146e-01
```


Notes

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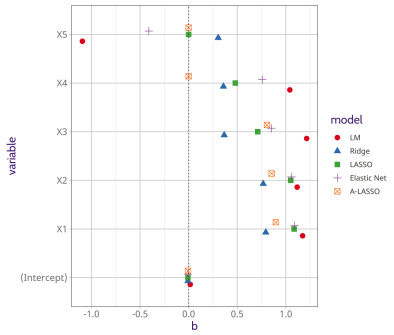
Adaptive LASSO and Collinearity

```
b.ridge <- coef(cv.glmnet(scale(coll$X),
                          coll$y, alpha=0))
# calculate weights
gamma <- 1
w <- 1/(abs(b.ridge)^gamma)
# estimate the LASSO with the weights
cval <- cv.glmnet(scale(coll$X),coll$y,
                  penalty.factor=w[-1])
```



Notes

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All predictions

```
r1 <- glmnet(X, y, alpha=0, lambda=rcv$lambda.mi
g1 <- glmnet(X, y, alpha=1, lambda=cvg$lambda.mi
g3 <- glmnet(X, y, alpha=s[which.min(cv.err)],
             lambda=cvg$lambda.min)
g4 <- glmnet(X, y, alpha=1, penalty.factor=w.ban
             lambda=cval$lambda.min)
yhat <- mod$fitted.values
names(yhat) <- NULL

preds <- tibble(
  ols=yhat,
  ridge = as.vector(predict(r1, newx=X)),
  lasso= as.vector(predict(g1, newx=X)),
  enet = as.vector(predict(g3, newx=X)),
  alasso = as.vector(predict(g4, newx=X)))
```

```
ggpairs(preds) + mytheme() + theme_bw()
```

Notes

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PGI Analysis

Table 2. Elastic net coefficients.

	Federal	Provincial	Regional
Intercept	0.000 [0.000, 0.000]	0.000 [−0.000, 0.000]	0.000 [−0.000, −0.000]
Eligible Voters (log)	−0.749 [−1.831, −0.126]	−0.002 [−0.027, −0.000]	−0.015 [−0.047, 0.000]
Average Age	0.000 [0.000, 0.000]	0.002 [0.000, 0.037]	0.000 [0.000, 0.000]
Median Income (log)	0.445 [0.000, 1.058]	−0.001 [−0.013, −0.000]	−0.510 [−0.566, −0.417]
% Unemployed	0.257 [0.000, 0.516]	0.002 [0.000, 0.037]	0.000 [0.000, 0.000]
% No Degree	0.001 [0.000, 0.297]	−0.002 [−0.035, −0.000]	0.000 [0.000, 0.000]
% University Degree	0.082 [−0.113, 0.183]	0.000 [−0.013, 0.000]	0.000 [0.000, 0.000]

Main entries are average coefficients across the cross-validation runs.
Entries in brackets show the range of coefficients across all 1000 cross-validation runs and are *not* confidence intervals.
In the results above, all variables were standardized to have mean 0 and unit variance.

Notes

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Inference After Selection

Inference gets much more complicated after model selection, given that variables are often selected *because* they are significant predictors. There are a few options for post-selection inference.

- Data Splitting - Split the sample into two halves - select on one set, test on the other. Most conservative (loss of power due to lower N).
- Data Carving - A small proportion of the sample is withheld from training and then the entire sample is used for testing (Fithian 2014).
- Exact post-selection inference possible for Forward Selection Regression and LASSO with fixed λ - `{SelectiveInferecen}` package in R (Tibshirani et al 2014).
- Valid post-selection inference for Linear LS Models - implemented in the `{PoSI}` package in R (Berk et al 2013)

Notes

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Variable Selection Methods: Cautions (1)

- If we have a very large number of predictors and we simply want a parsimonious predictive model, subset methods and the lasso could be really useful.
- When tackling collinearity, however, variable selection may results in a re-specified model that does not address the original research question (ridge regression could help).
- If the original model is correctly specified, then coefficient estimates following variable selection are *biased*. However, the bias may not be overwhelming if you started off with a severe collinearity problem

Notes

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Variable Selection Methods: Cautions (2)

- If our goal is to assess the individual predictors (or their relative impacts), variable selection models have serious implications
  - Standard errors calculated following variable selection overstate the precision of results - they do not control for relevant predictors and they do not account for model selection uncertainty.
  - A new sample may give different results, leading to inconsistent interpretation of "effects"
- These models, again, are really about *prediction* not hypothesis testing, though the can still be quite valuable.

Notes

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Using Regularization Techniques

- Smooth out otherwise complex functions.
- Use alternative methods to identify **important** variables.
- Select features that generate accurate predictions with lower variance.
  - Help solve collinearity problems.
- Theory testing? Not so much...

Notes

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